113 Class Problems: Euclidean Domains

1. Is the degree function a Euclidean function on $\mathbb{Z}[x]$? If so, give a proof. If not, give a counterexample.

   Solutions:

No. There do not exist $q(x), r(x) \in \mathbb{Z}[x]$ such that $x^2+1 = q(x)(2x+1) + r(x)$ where $\deg(r(x)) = 0$

not a unit in $\mathbb{Z}$

2. Let F be a field. Does F admit a Euclidean Function?

   Solutions:

Yes. $\varphi : F \setminus \{0\} \longrightarrow \mathbb{N} \cup \{0\}$

$a \longrightarrow 1$

Key point : $a, b \in F$, $b \neq 0 \Rightarrow a = (ab^{-1})b$

3. Using the Euclidean algorithm find a highest common factor of 20 and 13 in $\mathbb{Z}$. Again using the Euclidean algorithm, find $u, v \in \mathbb{Z}$ such that $2u + 13v$ is an HCF.

Solutions:

$$20 = 1 \cdot 13 + 7$$
$$13 = 1 \cdot 7 + 6$$
$$7 = 1 \cdot 6 + 1 \quad \leftarrow \text{HCF}(20, 13)$$
$$6 = 6 \cdot 1$$

$$7 = 1 \cdot 20 + (-1) \cdot 13$$
$$6 = 1 \cdot 13 + (-1) \cdot 7 = 1 \cdot 13 + (-1)(1 \cdot 20 + (-1) \cdot 13)$$
$$= 2 \cdot 13 + (-1) 20$$
$$1 = 1 \cdot 7 + (-1) \cdot 6 = 1 \cdot (1 \cdot 20 + (-1) \cdot 13) + (-1)(2 \cdot 13 + (-1) 20)$$
$$= 2 \cdot 20 + (-3) \cdot 13$$

4. Find a highest common factor of $x^4 + 3x - 1, x^3 + x^2 - 1 \in \mathbb{Q}[x]$. Hint: Do you remember how to do long division of polynomials?

Solutions:

$$\begin{array}{r} x - 1 \\ x^3 + x^2 - 1 \;\big)\; x^4 + 0 \cdot x^3 + 0 \cdot x^2 + 3x - 1 \\ -(x^4 + x^3 + 0x^2 - x + 0) \\ \hline -x^3 + 0x^2 + 4x - 1 \\ -(-x^3 - x^2 + 0x + 1) \\ \hline x^2 + 4x - 2 = r_1 \end{array}$$

$$\begin{array}{r} x - 3 \\ x^2 + 4x - 2 \;\big)\; x^3 + x^2 + 0x - 1 \\ -(x^3 + 4x^2 - 2x) \\ \hline -3x^2 + 2x - 1 \\ -(-3x^2 - 12x + 6) \\ \hline 14x - 7 = r_2 \end{array}$$

$$14x - 7 \nmid x^2 + 4x - 2$$

$$\Rightarrow r_3 \in \mathbb{Q}^*$$

$$\Rightarrow \text{HCF is } 1.$$